

Point Counting on Non-Hyperelliptic Genus 3 Curves with Automorphism Group $\mathbb{Z}/2\mathbb{Z}$ using Monsky-Washnitzer Cohomology

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ABSTRACT. We describe an algorithm to compute the zeta function of any non-hyperelliptic genus 3 plane curve C over a finite field with automorphism group $G = \mathbb{Z}/2\mathbb{Z}$. This algorithm computes in the Monsky-Washnitzer cohomology of the curve. Using the relation between the Monsky-Washnitzer cohomology of C and its quotient $E := C/G$, the computation splits into 2 parts: one in a subspace of the Monsky-Washnitzer cohomology and a second which reduces to the point counting on an elliptic curve E . The former corresponds to the dimension 2 abelian surface $\ker(\text{Jac}(C) \rightarrow E)$, on which we can compute with lower precision and with matrices of smaller dimension. Hence we obtain a faster algorithm than working directly on the curve C .

1. Introduction

Henn gave the table of the possible non-trivial groups which appear as automorphism groups of a non-hyperelliptic genus 3 curves, which can be found in Vermeulen's thesis [11]. The dimension of the set of moduli points of non-hyperelliptic genus 3 curves whose automorphism group contain $\mathbb{Z}/2\mathbb{Z}$ is 4 inside the moduli of genus 3 curves \mathcal{M}_3 of dimension 6. We thus obtain an algorithm to compute the zeta function of a large family of genus 3 curves.

In [5], Kedlaya used Monsky-Washnitzer cohomology to compute the zeta functions of hyperelliptic curves over finite fields. This method could be applied to general varieties, and there are already generalizations to superelliptic curves, $C_{a,b}$ curves and non-degenerate curves, see [1], [2] and [4]. This work also uses Monsky-Washnitzer cohomology but focuses on a smaller dimensional space associated to an abelian surface in the jacobian of C .

The remainder of this paper is organized as follows: Section 2 recalls the definition of Monsky-Washnitzer cohomology and results in this theory. In Section 3, we compute a basis of the cohomology and describe a way to do the reduction of a differential form. Reduction means to write a differential form as a linear combination of the basis. In particular, we give an upper bound of the denominator after a differential form is reduced. This bound makes the algorithm practical since it establishes a finite precision bound for the computation. Section 4 describes a

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way to compute a lift of Frobenius and Section 5 explains why the computation splits into 2 eigenspaces. Finally, Section 6 gives the algorithm and an analysis of its complexity.

2. Monsky-Washnitzer Cohomology

In this section, we recall the definition of Monsky-Washnitzer cohomology which is introduced by Monsky and Washnitzer in [7], [8], [9].

Monsky-Washnitzer cohomology is a p -adic cohomology theory defined for smooth affine varieties over finite fields. Let X be a smooth affine variety defined over a finite field $k := \mathbb{F}_q$ of characteristic p with coordinate ring \bar{A} which is a finitely generated k -algebra. In [3], Elkik showed that there exists a finitely generated smooth \mathbb{Z}_q -algebra A such that $A/pA \cong \bar{A}$, here \mathbb{Z}_q is the valuation ring of \mathbb{Q}_q , the degree $n := \log_p q$ unramified extension of \mathbb{Q}_p .

In general, A does not admit a lift of the Frobenius endomorphism \bar{F} on \bar{A} , but its p -adic completion A^∞ does. However, the dimension of the de Rham cohomology of A^∞ may be too big. For example, if $A = \mathbb{Z}_p[x]$, then $\sum_{n=0}^\infty p^n x^{p^n-1} dx$ is not an exact differential form since $\sum_{n=0}^\infty x^{p^n}$ is not in A^∞ , but each term of this sum is exact. The problem is that this differential form does not converge fast enough for its integral to converge as well.

Monsky and Washnitzer work with a subalgebra A^\dagger of A^∞ consisting of series which converge fast enough to solve the above problem. For

$$A = \mathbb{Z}_q[x_1, x_2, \dots, x_d]/(f_1, f_2, \dots, f_r),$$

the weak completion or dagger ring of A is

$$A^\dagger := \mathbb{Z}_q\langle x_1, x_2, \dots, x_d \rangle / (f_1, f_2, \dots, f_r)$$

where $\mathbb{Z}_q\langle x_1, x_2, \dots, x_d \rangle$ is the subring of A^∞ which consists of overconvergent power series

$$\left\{ \sum_{\alpha} a_{\alpha} x^{\alpha} \in \mathbb{Z}_q[[x_1, x_2, \dots, x_d]] \mid \liminf_{|\alpha| \rightarrow \infty} \frac{v_p(a_{\alpha})}{|\alpha|} > 0 \right\}$$

with $\alpha := (\alpha_1, \dots, \alpha_d)$, $x^{\alpha} := x_1^{\alpha_1} \dots x_d^{\alpha_d}$, $|\alpha| = \sum_{i=1}^d \alpha_i$ and v_p is the usual p -adic valuation on \mathbb{Z}_q .

DEFINITION 2.1. The Monsky-Washnitzer cohomology of X/\mathbb{F}_q is the de Rham cohomology of $A^\dagger \otimes_{\mathbb{Z}_q} \mathbb{Q}_q$. More precisely, let $D^0(A^\dagger) := A^\dagger$, $D^1(A^\dagger)$ be its universal module of differentials

$$D^1(A^\dagger) := (A^\dagger dx_1 + \dots + A^\dagger dx_d) / \left(\sum_{j=1}^r A^\dagger \left(\frac{\partial f_j}{\partial x_1} dx_1 + \dots + \frac{\partial f_j}{\partial x_d} dx_d \right) \right)$$

and $D^i(A^\dagger)$ be the i -th exterior product of $D^1(A^\dagger)$. Let $H^i(\bar{A}, \mathbb{Z}_q)$ be the i -th cohomology group of the complex

$$0 \rightarrow D^0(A^\dagger) \xrightarrow{d_0} D^1(A^\dagger) \xrightarrow{d_1} D^2(A^\dagger) \xrightarrow{d_2} D^3(A^\dagger) \xrightarrow{d_3} \dots$$

where d_i is the usual differentiation. Then the i -th Monsky-Washnitzer cohomology group of X (or of \bar{A}) is $H^i(\bar{A}, \mathbb{Z}_q) \otimes_{\mathbb{Z}_q} \mathbb{Q}_q$, which is denoted by $H^i_{MW}(X/\mathbb{F}_q)$ (or $H^i(\bar{A}, \mathbb{Q}_q)$).

The Monsky-Washnitzer cohomology has the following properties, see [10].

THEOREM 2.2. *For a smooth finitely generated \mathbb{F}_q -algebra \overline{A} , we have*

- (a) *The map $\overline{A} \mapsto H^i(\overline{A}, \mathbb{Q}_q)$ is well defined and functorial.*
- (b) *There exists a \mathbb{Z}_q -algebra homomorphism $F_q : A^\dagger \rightarrow A^\ddagger$ which lifts the Frobenius endomorphism of \overline{A} . Furthermore, any two lifts induce homotopic maps on the complex $D^i(A^\dagger)$. Hence they induce the same map $F_{q,*} : H^i(\overline{A}, \mathbb{Q}_q) \rightarrow H^i(\overline{A}, \mathbb{Q}_q)$ on the Monsky-Washnitzer cohomology.*

The following Lefschetz fixed point formula allows us to compute the zeta function of $X = \text{Spec}(\overline{A})$ using Monsky-Washnitzer cohomology.

THEOREM 2.3 (Lefschetz fixed point formula). *Let X/\mathbb{F}_q be a smooth affine variety of dimension d . Then we have*

$$X(\mathbb{F}_q) = \sum_{i=0}^d (-1)^i \text{Tr} \left(q^d F_{q,*}^{-1} \Big| H_{MW}^i(X/\mathbb{F}_q) \right).$$

3. Cohomology of Non-Hyperelliptic Genus 3 Plane Curves with Automorphism Group $\mathbb{Z}/2\mathbb{Z}$

In this article, we consider non-hyperelliptic smooth projective plane curves C of genus 3 whose automorphism group contains $\mathbb{Z}/2\mathbb{Z}$ over a finite field \mathbb{F}_q of characteristic $p \neq 2$. Such curves can be written (up to isomorphism) as

$$C : \overline{F} := Y^4 + \overline{G}(X, Z)X^2 + \overline{H}(X, Z) = 0,$$

with $\overline{G}(X, Z)$ and $\overline{H}(X, Z) \in \mathbb{F}_q[X, Z]$ which are homogeneous of degree 2 and 4 respectively. We assume that C is smooth. Since the Monsky-Washnitzer cohomology is defined for smooth affine varieties, we also consider the affine part of C

$$C_{\text{aff}} : \overline{f} := y^4 + \overline{g}(x)y^2 + \overline{h}(x) = 0,$$

where $\overline{g}(x) = \overline{G}(x, 1)$ and $\overline{h}(x) = \overline{H}(x, 1)$ are the dehomogenizations of $\overline{G}(X, Z)$ and $\overline{H}(X, Z)$ with respect to Z . In this section, we compute the Monsky-Washnitzer cohomology $H_{MW}^i(C_{\text{aff}}/\mathbb{F}_q)$ of C_{aff} and relate the zeta function of C/\mathbb{F}_q to the characteristic polynomial of the Frobenius action $F_{q,*}$ on $H_{MW}^i(C_{\text{aff}}/\mathbb{F}_q)$.

Choose arbitrary lifts $G(X, Z)$ and $H(X, Z) \in \mathbb{Z}_q[X, Z]$ of $\overline{G}(X, Z)$ and $\overline{H}(X, Z)$ such that $\deg_X G = \deg_X \overline{G}$ and $\deg_X H = \deg_X \overline{H}$. Let $g(x) := G(x, 1)$ and $h(x) := H(x, 1)$ be the dehomogenizations. Consider the following two curves

$$\tilde{C} : F := Y^4 + G(X, Z)Y^2 + H(X, Z) = 0,$$

and

$$(3.1) \quad \tilde{C}_{\text{aff}} : f := y^4 + g(x)y^2 + h(x) = 0$$

Since the reduction of F modulo the maximal ideal (p) of \mathbb{Z}_q is \overline{F} which defines a smooth projective curve C , the generic fiber $\tilde{C}_\xi := \tilde{C} \times_{\mathbb{Z}_q} \mathbb{Q}_q$ of \tilde{C} is also smooth. Using the three facts that the reduction of f modulo the maximal ideal (p) of \mathbb{Z}_q equals to \overline{f} which is not zero in $\mathbb{F}_q[x, y]$, that $\overline{A} := \mathbb{F}_q[x, y]/(\overline{f}(x, y))$ is an integral domain and that p is a prime element in the unique factorization domain $\mathbb{Z}_q[x, y]$, one sees that A is an integral domain and hence it is flat over \mathbb{Z}_q . This shows that A is a finitely generated smooth \mathbb{Z}_q -algebra, so we can work with A to apply the theory of Monsky-Washnitzer cohomology. The above arguments also show that the generic fiber \tilde{C}_ξ of \tilde{C} is a geometrically integral smooth projective curve over \mathbb{Q}_q .

Although we can compute the Monsky-Washnitzer cohomology of the affine curve $C_{\text{aff}}/\mathbb{F}_q$ by explicit reduction algorithms and the control of denominators, we use the following theorem instead, see [6], and compute the algebraic de Rham cohomology $H_{\text{dR}}^i(\tilde{C}_{\xi,\text{aff}}/\mathbb{Q}_q)$ of the curve $\tilde{C}_{\xi,\text{aff}}/\mathbb{Q}_q$, the affine part of the generical fiber \tilde{C}_{ξ} . Note that we are concerned with curves, hence the divisors are always normal crossings.

THEOREM 3.1. *Let Y/\mathbb{Z}_q be a smooth proper scheme, Z be a relative normal crossings divisor and $X := Y \setminus Z$ is affine. Then there is a canonical isomorphism*

$$H_{\text{dR}}^i(X_{\xi}/\mathbb{Q}_q) \rightarrow H_{\text{MW}}^i(X_p/\mathbb{F}_q),$$

where X_{ξ} is the generic fiber and X_p is the special fiber of X/\mathbb{Z}_q , namely, the fibers of X at the closed point (p) of $\text{Spec}(\mathbb{Z}_q)$.

3.1. Geometry. Before we compute the algebraic de Rham cohomology of the affine curve $\tilde{C}_{\xi,\text{aff}}$, we need to study its geometry. The coordinate ring of $\tilde{C}_{\xi,\text{aff}}$ is $A := \mathbb{Q}_q[x, y]/(f(x, y))$, here $f(x, y) = y^4 + g(x)y^2 + h(x) \in \mathbb{Z}_q[x, y]$ with $\deg(g) \leq 2$ and $\deg(h) \leq 4$. We write $g(x)$ and $h(x)$ as following

$$(3.2) \quad \begin{aligned} g(x) &= a_2x^2 + a_1x + a_0, \quad a_i \in \mathbb{Z}_q \\ h(x) &= b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0, \quad b_i \in \mathbb{Z}_q \end{aligned}$$

There are four cases to consider:

Case 1. $b_4 = 0$ and $a_2 = 0$

There is only one point at infinity which is $P_{\infty} := (1 : 0 : 0)$. Using the fact that \tilde{C}_{ξ} is smooth at P_{∞} , one shows that $b_3 \neq 0$. The dimension of the first algebraic de Rham cohomology of $\tilde{C}_{\xi,\text{aff}}/\mathbb{Q}_q$ is $2g + N_{\infty} - 1 = 2 \cdot 3 + 1 - 1 = 6$, here g is the genus of \tilde{C}_{ξ} , which equals to the genus of C , and N_{∞} is the number of points at infinity. We have $v_{P_{\infty}}(x) = -4$ and $v_{P_{\infty}}(y) = -3$. The local parameter at P_{∞} is $t := b_3x^2/y^3$. The expansions of x and y as Laurent series of the local parameter t are $x = -b_3/t^4 + \dots$ and $y = b_3/t^3 + \dots$.

Case 2. $b_4 = 0$ and $a_2 \neq 0$

There are 3 points at infinity: $P_{\infty} := (1 : 0 : 0)$ and $P_{\infty,\pm} := (1 : \pm\alpha : 0)$ with $\alpha^2 = -a_2$. Using the fact that \tilde{C}_{ξ} is smooth at P_{∞} , one shows that $b_3 \neq 0$. (The condition $a_2 \neq 0$ implies the smoothness at $P_{\infty,+}$ and $P_{\infty,-}$.) We have $\dim_{\mathbb{Q}_q} H_{\text{dR}}^1(\tilde{C}_{\xi,\text{aff}}/\mathbb{Q}_q) = 2 \cdot 3 + 3 - 1 = 8$, $v_{P_{\infty,\pm}}(x) = v_{P_{\infty,\pm}}(y) = -1$, $v_{P_{\infty}}(x) = -2$ and $v_{P_{\infty}}(y) = -1$. The local parameters at P_{∞} and $P_{\infty,\pm}$ are $t := 1/y$ and $t_{\pm} := 1/x$. The expansions of x and y at P_{∞} and $P_{\infty,\pm}$ as Laurent series of the local parameters are $x = \beta/t^2 + \gamma + \delta t^2 + \dots$ with $\beta = -a_2/b_3$, $y = 1/t$, and $x = 1/t_{\pm}$, $y = \pm\alpha/t + \dots$.

Case 3. $b_4 \neq 0$ and $a_2^2 - 4b_4 = 0$

There are 2 points at infinity: $P_{\infty,\pm} := (1 : \pm\alpha : 0)$ with $\alpha = (-a_2/2)^{1/2}$. Using the fact that \tilde{C}_{ξ} is smooth at $P_{\infty,+}$, one shows that $a_1a_2 - 2b_3 \neq 0$. We have $\dim_{\mathbb{Q}_q} H_{\text{dR}}^1(\tilde{C}_{\xi,\text{aff}}/\mathbb{Q}_q) = 2 \cdot 3 + 2 - 1 = 7$ and $v_{P_{\infty,\pm}}(x) = v_{P_{\infty,\pm}}(y) = -2$. The local parameters are $t_{\pm} := y/x \mp \alpha$. The expansions of x and y as Laurent series of the local parameters are $x = \beta/t_{\pm}^2 + \gamma/t_{\pm} + \dots$ and $y = \pm\alpha\beta/t_{\pm}^2 + (\beta \pm \alpha\gamma)/t_{\pm} + \dots$ with $\beta = -(a_1a_2 - 2b_3)/4a_2$ and $\gamma = \alpha(a_1a_2 + 2b_3)/2a_2^2$.

Case 4. $b_4 \neq 0$ and $a_2^2 - 4b_4 \neq 0$

There are 4 points at infinity which are $P_{\infty,i,\pm} := (1 : \pm\alpha_i : 0)$ with $\pm\alpha_1$ and

$\pm\alpha_2$ are the four roots of $y^4 + a_2y^2 + b_4 = 0$. We have $\dim_{\mathbb{Q}_q} H_{dR}^1(\tilde{C}_{\xi, \text{aff}}/\mathbb{Q}_q) = 2 \cdot 3 + 4 - 1 = 9$ and $v_{P_{\infty, i, \pm}}(x) = v_{P_{\infty, i, \pm}}(y) = -1$. The local parameters are $t := 1/x$. The expansions of x and y as Laurent series of t are $x = 1/t$ and $y = \pm\alpha_i/t + \dots$.

In order to analyze the control of denominators later, we need to impose further assumptions on the choice of the lift $F(X, Y, Z)$.

ASSUMPTION 3.2. The coefficients a_i and b_j of $g(x)$ and $h(x)$ in (3.2) are either 0 or units in \mathbb{Z}_q . Furthermore, $a_2^2 - 4b_4$ is either 0 or a unit in \mathbb{Z}_q .

REMARK 3.3. A lift which satisfies the above assumptions could be constructed by using Teichmüller lift. The choice of such a lift is to preserve the geometric structure. Under these assumptions, we introduce some facts which will be used later. The expansions of x and y as Laurent series of the local parameters have integral coefficients.¹ This means that x and y are in $\mathcal{O}(t)$, here \mathcal{O} is the integral closure of \mathbb{Z}_q in a finite extension $\mathbb{Q}_q(\alpha)$ ($\mathbb{Q}_q(\alpha_1, \alpha_2)$ in Case 4) of \mathbb{Q}_q and α is the Y -coordinate of the points at infinity discussed above. Furthermore, the coefficients of the lowest terms² in these Laurent series are units in \mathcal{O} . (In Case 3, one shows that $a_1a_2 - 2b_3$ is a unit in \mathbb{Z}_q by using C is smooth.) \tilde{C}_{ξ} and \bar{C} have the same geometry. In Case 4, $\sqrt{a_2^2 - 4b_4}$ is a unit in \mathcal{O} .

3.2. The Reduction Algorithm and Algebraic de Rham Cohomology. In this subsection, we present the reduction algorithm and use it to compute $H_{dR}^1(\tilde{C}_{\xi, \text{aff}}/\mathbb{Q}_q)$. First of all, since $(i + 1)x^jy^i dy = d(x^jy^{i+1}) - jx^{j-1}y^{i+1} dx$ for all $i \geq 0$, the universal module of differential Ω^1 of $\tilde{C}_{\xi, \text{aff}}/\mathbb{Q}_q$ is generated by $\{x^jy^i dx \mid i \geq 0, j \geq 0\}$. From the defining equation (3.1), Ω^1 is generated by $\{x^jy^i dx \mid 1 \leq i \leq 3, j \geq 0\}$. Since

$$0 = df = d(y^4 + g(x)y^2 + h(x)) = (g'(x)y^2 + h'(x)) dx + (4y^3 + 2g(x)y) dy,$$

we have

$$x^k (g'(x)y^2 + h'(x)) y^l dx + x^k (4y^3 + 2g(x)y) y^l dy = 0.$$

Combine with the following equation

$$\begin{aligned} & d\left(x^k \left(\frac{4}{l+4}y^{l+4} + \frac{2}{l+2}g(x)y^{l+2}\right)\right) - x^k (4y^3 + 2g(x)y) y^l dy \\ &= \left(\frac{4}{l+4}kx^{k-1}y^{l+4} + \frac{2}{l+2}(kx^{k-1}g(x) + x^k g'(x)) y^{l+2}\right) dx, \end{aligned}$$

one gets

$$\begin{aligned} & x^k \left(\frac{l}{l+2}g'(x)y^2 + h'(x)\right) y^l dx - kx^{k-1} \left(\frac{4}{l+4}y^4 + \frac{2}{l+2}g(x)y^2\right) y^l dx \\ &= d(S_{l,k}), \text{ where } S_{l,k} := -x^k \left(\frac{4}{l+4}y^{l+4} + \frac{2}{l+2}g(x)y^{l+2}\right) \end{aligned}$$

Using $y^4 = -(g(x)y^2 + h(x))$ and the explicit description of $g(x)$ and $h(x)$ in (3.2), the above equation becomes

$$(3.3) \quad \left(\sum_{j=-1}^3 \Gamma_{k,l,0,j} x^{k+j} y^j + \sum_{j=-1}^1 \Gamma_{k,l,2,j} x^{k+j} y^{l+2}\right) dx = d(S_{l,k}),$$

¹Use Hensel's lemma. For Case 1, one needs a 2-variable version of Hensel's lemma.

²In particular, $\alpha, \beta \in \mathcal{O}^*$ if we use the notations in the above classification.

here the coefficients $\Gamma_{k,l,0,j}$ and $\Gamma_{k,l,2,j}$ are defined as following

$$\Gamma_{k,l,0,j} := (j + 1 + \frac{4k}{l+4})b_{j+1}, \quad \Gamma_{k,l,2,j} := \frac{l}{l+2}(j + 1 + \frac{2k}{l+4})a_{j+1}.$$

In order to make things more clear, we use the following notation:

DEFINITION 3.4. A family of matrices M_k of size $m \times n$ with entries $(M_k)_{i,j} = m_{k,i,j} \in \mathbb{Q}_q$ is called a family of reduction matrices if

$$\left(\sum_{i=1}^m \sum_{j=1}^n m_{k,i,j} x^{k+j-3} y^{i-1} \right) dx \equiv 0 \text{ in } \Omega^1, \text{ for all } k.$$

For reduction matrices M_k , we define $M_k dx$ to be

$$\left(\sum_{i=1}^m \sum_{j=1}^n m_{k,i,j} x^{k+j-3} y^{i-1} \right) dx.$$

For example, from (3.3), we have reduction matrices of size $(l + 3) \times 6$ which has non-zero entries only at the $(l + 1)$ -th and $(l + 3)$ -th rows

$$M_{l,k}^0 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Gamma_{l,k,0,-1} & \Gamma_{l,k,0,0} & \Gamma_{l,k,0,1} & \Gamma_{l,k,0,2} & \Gamma_{l,k,0,3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Gamma_{l,k,2,-1} & \Gamma_{l,k,2,0} & \Gamma_{l,k,2,1} & 0 & 0 \end{pmatrix}$$

and $M_{l,k}^0 dx = d(S_{l,k})$. The superscript 0 that appears in $M_{l,k}^0$ means that it is obtained from (3.3), without further reduction.

We have to consider $l = 1, 2$ and 3 , which give the reductions of $x^{k+3}y^2 dx$ or $x^{k+2}y^2 dx$:

$l = 1$.

As mentioned above, we have reduction matrices of size 4×6

$$M_{1,k} = \frac{1}{15} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12kb_0 & (12k + 15)b_1 & (12k + 30)b_2 & (12k + 45)b_3 & (12k + 60)b_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2ka_0 & (2k + 5)a_1 & (2k + 10)a_2 & 0 & 0 \end{pmatrix}$$

From 3.1, we know that one of b_4 and b_3 is non-zero. Hence $x^{k+3}y dx$ or $x^{k+2}y dx$ can be reduced to a linear combination of $\{x^j y^i dx\}$ with $i = 1$ or 3 and $j \leq k + 2$ (if $b_4 \neq 0$) or $j \leq k + 1$ (if $b_4 = 0$), which have smaller degree in x .

$l = 2$.

From (3.2), we have reduction matrices of size 5×6 , so y^4 is involved. Using the defining equation (3.1) to reduce the degree in y , one gets a reduction matrix $M_{2,k}$ whose transpose is

$$M_{2,k}^t = -\frac{1}{6} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k(a_0^2 - 4b_0) & 0 & 0 \\ 0 & 0 & (2k + 3)(a_0 a_1 - 2b_1) & 0 & 0 \\ 0 & 0 & (k + 3)(a_1^2 + 2a_0 a_2 - 4b_2) & 0 & 0 \\ 0 & 0 & (2k + 9)(a_1 a_2 - 2b_3) & 0 & 0 \\ 0 & 0 & (k + 6)(a_2^2 - 4b_4) & 0 & 0 \end{pmatrix}$$

This gives us the reductions of $x^{k+3}y^2dx$ or $x^{k+2}y^2dx$ depending on the nullity of $a_2^2 - 4b_4$.

$l = 3$.

As at the start of $l = 2$ (dealing with y^5), one gets $M_{3,k}^1$ whose transpose is

$$-\frac{1}{35} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 6ka_0b_0 & 0 & k(6a_0^2 - 20b_0) \\ 0 & 6k(a_1b_0 + a_0b_1) + 21a_1b_0 & 0 & (4k + 7)(3a_0a_1 - 5b_1) \\ 0 & 6k(a_2b_0 + a_1b_1 + a_0b_2) + 21a_1b_1 + 42a_2b_0 & 0 & (2k + 7)(3a_1^2 + 6a_0a_2 - 10b_2) \\ 0 & 6k(a_2b_1 + a_1b_2 + a_0b_3) + 21a_1b_2 + 42a_2b_1 & 0 & (4k + 21)(3a_1a_2 - 5b_3) \\ 0 & 6k(a_2b_2 + a_1b_3 + a_0b_4) + 21a_1b_3 + 42a_2b_2 & 0 & (2k + 14)(3a_2^2 - 10b_4) \\ 0 & (6k + 21)(a_2b_3 + a_1b_4) + 21a_2b_3 & 0 & 0 \\ 0 & (6k + 42)a_2b_4 & 0 & 0 \end{pmatrix}$$

Since we want to reduce x^jy^3dx to those with smaller degree in x , $M_{3,k}^1$ is not suitable since it has (possible) non-zero entries which correspond to $x^{k+5}y^1$, $x^{k+4}y^1$ and $x^{k+3}y^1$. We use $l = 1$ to reduce $x^{k+3}ydx$ to $\{x^{j_1}ydx, x^{j_2}y^3dx \mid j_1 \leq k + 2, j_2 \leq k + 1\}$ if $b_4 \neq 0$, or reduce $x^{k+2}ydx$ to $\{x^{j_1}ydx, x^{j_2}y^3dx \mid j_1 \leq k + 1, j_2 \leq k + 1\}$ if $b_4 = 0$. Then use this result to reduce $x^{k+3}ydx$ (if $b_4 = 0$), $x^{k+4}ydx$ and $x^{k+5}ydx$ (if $b_4 \neq 0$) iteratively to $\{x^{j_1}ydx, x^{j_2}y^3dx \mid j_1 \leq k + 2, j_2 \leq k + 3\}$, depending the nullity of b_4 . Finally, use these reductions in $M_{3,k}^1$, we get reduction matrices in Case 3 and Case 4: $b_4 \neq 0$

$$M'_{3,k} = c \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & *_{2,5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & *_{4,5} & 384(k + 4)(k + 5)(k + 6)(k + 7)b_4^3(a_2^2 - 4b_4) \end{pmatrix}$$

with $c = -1 / (2688(k + 4)(k + 5)(k + 6)b_4^3)$ and

$$*_{4,5} = 96b_4^2(k + 4)(k + 5)(k + 6) ((8k + 44)a_1a_2b_4 - a_2^2b_3 - (16k + 84)b_3b_4)$$

which equals to $384b_4^3(k + 4)(k + 5)(k + 6)(2k + 11)(a_1a_2 - 2b_3)$ if $a_2^2 - 4b_4 = 0$, and

$$*_{2,5} = 36(k + 4)b_4 \left((6k^2 + 136k + 285)a_2b_3^3 - 8(6k^2 + 56k + 125)a_2b_2b_3 + 48(k^2 + 11k + 30)a_2b_1b_4^2 + 16(k^2 + 6k)a_0b_3b_4^2 - 2(8k^2 + 58k + 105)a_1b_3^2b_4 + 16(2k^2 + 17k + 35)a_1b_2b_4^2 \right)$$

Notice that $*_{2,5}$ may be non-zero, but it corresponds to $x^{k+2}y$ and since $b_4 \neq 0$, one can use $M_{1,k-1}$ in the case $l = 1$ to reduce $x^{k+2}ydx$ and get a new reduction matrices

$$M_{3,k} = c \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & *_{4,5} & 384(k + 4)(k + 5)(k + 6)(k + 7)b_4^3(a_2^2 - 4b_4) \end{pmatrix}$$

Notice that the reduction of $x^{k+2}ydx$ using $M_{1,k-1}$ only involve x^jydx with $k - 2 \leq j \leq k + 1$ and x^jy^3 with $k - 2 \leq j \leq k$, the last two columns of $M_{3,k}$ and $M_{3,k}^2$ are the same except the (2, 5)-entry, and $M_{3,k}$ satisfies the condition in Definition 3.4, so they are indeed reduction matrices. The reduction involves division by $12(k + 4)b_4$, hence $c^{-1}M_{3,k}$ has integral coefficients.

Case 1 and Case 2: $b_4 = 0$

$$M_{3,k} = c \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & *_{4,5} & 2(k+7)(2k+11)(4k+15)(4k+19)b_3^2 a_2^2 \end{pmatrix}$$

with $c = -1/(7(4k+15)(4k+19)(4k+23)b_3^2)$ and

$$*_{4,5} = (4k+15)b_3 \left((32k^3 + 504k^2 + 2648k + 4641)a_1 a_2 b_3 - (4k^2 + 52k + 168)a_2^2 b_2 - (64k^3 + 1008k^2 + 5276k + 9177)b_3^2 \right)$$

which equals to $-(4k+15)(4k+19)(4k+21)(4k+23)b_3^2$ if $a_2 = 0$.

Now we can compute the algebraic de Rham cohomology $H_{dR}^1(\tilde{C}_{\xi, \text{aff}}/\mathbb{Q}_q)$ of $\tilde{C}_{\xi, \text{aff}}/\mathbb{Q}_q$.

PROPOSITION 3.5. *The algebraic de Rham cohomology $H_{dR}^1(\tilde{C}_{\xi, \text{aff}}/\mathbb{Q}_q)$ has a basis*

- (1) $\{ydx, y^2dx, y^3dx, xydx, xy^2dx, xy^3dx\}$, if $b_4 = 0$ and $a_2 = 0$.
- (2) $\{ydx, y^2dx, y^3dx, xydx, xy^2dx, xy^3dx, x^2y^2dx, x^2y^3dx\}$, if $b_4 = 0$ and $a_2 \neq 0$.
- (3) $\{ydx, y^2dx, y^3dx, xydx, xy^2dx, xy^3dx, x^2ydx\}$, if $b_4 \neq 0$ and $a_2^2 - 4b_4 = 0$.
- (4) $\{ydx, y^2dx, y^3dx, xydx, xy^2dx, xy^3dx, x^2ydx, x^2y^2dx, x^2y^3dx\}$, if $b_4 \neq 0$ and $a_2^2 - 4b_4 \neq 0$.

PROOF. We give the proof for (c). For other parts, the proofs are all similar. Suppose $b_4 \neq 0$ and $a_2^2 - 4b_4 = 0$. The reduction matrices $M_{1,k}$ shows that $x^{k+3}ydx$ is a linear combination of $\{x^{j_1}ydx, x^{j_2}y^3dx \mid k-1 \leq j_1 \leq k+2, k-1 \leq j_2 \leq k+1\}$, since $b_4 \neq 0$. So each x^jydx with $j \geq 3$ can be reduced. The (3, 6)-entry of the reduction matrix $M_{2,k}$ is $(k+6)(a_2^2 - 4b_4) = 0$, but its (3, 5)-entry is $(2k+9)(a_1 a_2 - 2b_3)$ which is non-zero by Case 3 in Section 3.1. So $x^{k+2}y^2dx$ is a linear combination of $\{x^jy^2 \mid k-1 \leq j \leq k+1\}$ and hence each x^jy^2dx with $j \geq 2$ can be reduced. The reduction matrix $M_{3,k}$ in Case 3 has $(k+7)(a_2^2 - 4b_4) = 0$ at the (4, 6)-entry, but its (4, 5)-entry is $*_{4,5} = (2k+11)(a_1 a_2 - 2b_3) \neq 0$. So $x^{k+2}y^3dx$ is a linear combination of $\{x^{j_1}ydx, x^{j_2}y^3dx \mid k-2 \leq j_1 \leq k+1, k-2 \leq j_2 \leq k+1\}$ and hence each x^jy^3 with $j \geq 2$ can be reduced. This completes the proof for (c). \square

The following table give a more clear description of these basis.

<table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>dx</td><td>1</td><td>x</td><td>x^2</td></tr> <tr><td>y</td><td>•</td><td>•</td><td>×</td></tr> <tr><td>y^2</td><td>•</td><td>•</td><td>×</td></tr> <tr><td>y^3</td><td>•</td><td>•</td><td>×</td></tr> </table>	dx	1	x	x^2	y	•	•	×	y^2	•	•	×	y^3	•	•	×	<table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>dx</td><td>1</td><td>x</td><td>x^2</td></tr> <tr><td>y</td><td>•</td><td>•</td><td>×</td></tr> <tr><td>y^2</td><td>•</td><td>•</td><td>•</td></tr> <tr><td>y^3</td><td>•</td><td>•</td><td>•</td></tr> </table>	dx	1	x	x^2	y	•	•	×	y^2	•	•	•	y^3	•	•	•	<table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>dx</td><td>1</td><td>x</td><td>x^2</td></tr> <tr><td>y</td><td>•</td><td>•</td><td>•</td></tr> <tr><td>y^2</td><td>•</td><td>•</td><td>×</td></tr> <tr><td>y^3</td><td>•</td><td>•</td><td>×</td></tr> </table>	dx	1	x	x^2	y	•	•	•	y^2	•	•	×	y^3	•	•	×	<table border="1" style="border-collapse: collapse; text-align: center;"> <tr><td>dx</td><td>1</td><td>x</td><td>x^2</td></tr> <tr><td>y</td><td>•</td><td>•</td><td>•</td></tr> <tr><td>y^2</td><td>•</td><td>•</td><td>•</td></tr> <tr><td>y^3</td><td>•</td><td>•</td><td>•</td></tr> </table>	dx	1	x	x^2	y	•	•	•	y^2	•	•	•	y^3	•	•	•
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Case 1	Case 2	Case 3	Case 4																																																																

TABLE 1. basis of $H_{dR}^1(\tilde{C}_{\xi, \text{aff}}/\mathbb{Q}_q)$

3.3. Control of the denominators in the reduction algorithm and Monsky-Washnitzer cohomology. The reduction algorithm in subsection 3.2 allows us to obtain a basis of $H_{dR}^1(\tilde{C}_{\xi, \text{aff}}/\mathbb{Q}_q)$. By Theorem 3.1, this basis also forms a basis of the Monsky-Washnitzer cohomology $H_{MW}^1(C_{\text{aff}}/\mathbb{F}_q)$. One can also prove this by the following upper bound on the denominators that appear during the reduction process. This bound provides the precision necessary for our algorithm.

Before stating the main result of this subsection, we fix some notations. For a local parameter t at a point at infinity P_∞ , we write the Laurent series expansion of x , y and $x^j y^i$ with respect to t as following:

$$(3.4) \quad x = \sum_{s=v_p(x)}^{\infty} \delta_s^{0,1} t^s, \quad y = \sum_{s=v_p(y)}^{\infty} \delta_s^{1,0} t^s \quad \text{and} \quad x^j y^i = \sum_{s=v_p(x^j y^i)}^{\infty} \delta_s^{i,j} t^s.$$

If a subscript is used to denote a local parameter at some point, we use this subscript in the coefficients of the above expansion. For example, in Case 3, we write $x^j y^i = \sum_s^{\infty} \delta_{s,+}^{i,j} t_+^s$ at $P_{\infty,+}$ and $x^j y^i = \sum_s^{\infty} \delta_{s,-}^{i,j} t_-^s$ at $P_{\infty,-}$. Recall that all the coefficients $\delta_s^{i,j}$ are in \mathcal{O} in all cases that we are concerned, see Remark 3.3.

PROPOSITION 3.6. *Write*

$$(3.5) \quad x^k y^l dx = \sum_{i=1}^3 \sum_{j=0}^2 a_{i,j} x^j y^i dx + dS, \quad \text{where} \quad S = \sum_{i=0}^3 \sum_{j \geq 0} b_{i,j} x^j y^i$$

with $a_{i,j}$ and $b_{i,j} \in \mathbb{Q}_q$, $a_{i,j} = 0$ if $x^j y^i$ is not in the basis in Proposition 3.5, $1 \leq l \leq 3$ and $k \in \mathbb{N}$. Then

- (a) One can choose S with the property that $b_{i,j} = 0$ if $i - l \not\equiv 0 \pmod{2}$.
- (b) For any S in (3.5) which satisfies the property in (a), we have $b_{i,j-i} = 0$ for all $0 \leq i \leq 3$ and $j \geq k+5$. Furthermore $p^m b_{i,j-i} \in \mathcal{O}$ for all $0 \leq i \leq 3$ and $j \geq 7$, where $m = \lfloor \log_p(4k+8) \rfloor$.
- (c) $p^{m+\Delta+1} a_{i,j} \in \mathbb{Z}_q$, where $\Delta := 11 (\lfloor \log_p(63) \rfloor + \tau(p))$ with $\tau(3) = 5, \tau(5) = 3, \tau(p) = 1$ for $p = 7, 11, 13$ and $\tau(p) = 0$ if $p > 13$.

PROOF. (a) Using the automorphism $y \rightarrow -y$, or by a direct analysis on the reduction process discussed in the previous subsection. (b) We prove this for Case 4, proofs for other cases are all similar. From the expansions in Case 4, one obtains

$$(3.6) \quad x^j y^i = \sum_{s=-(i+j)}^{\infty} \delta_{s,\mu,\pm}^{i,j} t_{\mu,\pm}^s$$

with $\delta_{-(i+j),\mu,\pm}^{i,j} = (\pm\alpha_\mu)^i$ and $\mu = 1, 2$. There is an integer $M > 0$ such that $b_{i,j} = 0$ for all $j > M$. Hence

$$\begin{aligned} S &= \sum_{i=0}^3 \sum_{j \geq 0} b_{i,j} x^j y^i = \sum_{i=0}^3 \sum_{j \geq 0} b_{i,j} \sum_{s=-(i+j)}^{\infty} \delta_{s,\mu,\pm}^{i,j} t_{\mu,\pm}^s \\ &= \sum_{j=0}^{M+3} \left(\sum_{i=0}^3 b_{i,j-i} \delta_{-j,\mu,\pm}^{i,j-i} + \sum_{j' > j}^{M+3} \left(\sum_{i=0}^3 b_{i,j'-i} \delta_{-j,\mu,\pm}^{i,j'-i} \right) \right) t_{\mu,\pm}^{-j} \end{aligned}$$

Since $v_{p_\infty, \mu, \pm}(a_{i,j}x^jy^i dx) \geq 7$ and $v_{p_\infty, \mu, \pm}(x^k y^l dx) \geq -(k+5)$ and the expansions of $x^k y^l dx$ have integral coefficients, we have

$$(3.7) \quad j \cdot \left(\sum_{i=0}^3 b_{i,j-i} \delta_{-j, \mu, \pm}^{i,j-i} + \sum_{j' > j}^{M+3} \left(\sum_{i=0}^3 b_{i,j'-i} \delta_{-j, \mu, \pm}^{i,j'-i} \right) \right) \in \mathcal{O}$$

for all $j \geq 7$ and it is zero if $j \geq k+5$. Combine (3.7) with the property in (a) and the fact that α_1 and α_2 are units in \mathcal{O} (Fact 3.3), we get

$$j \cdot \left(1 \cdot b_{i,j-i} + \alpha_\mu^2 \cdot b_{i+2,j-i-2} + \sum_{j' > j}^{M+3} \sum_{i=0}^3 * \right) \in \mathcal{O}$$

for $i = 0, 1, j \geq 7$ and $\mu = 1, 2$, and it is zero if $j \geq k+5$. Since $\alpha_1^2 - \alpha_2^2 = \sqrt{D}$, here $D = a_2^2 - 4b_4 \neq 0$, one obtains

$$(3.8) \quad j \cdot \left(1 \cdot b_{i,j-i} + \sqrt{D} \cdot b_{i+2,j-i-2} + \sum_{j' > j}^{M+3} \sum_{i=0}^3 * \right) \in \mathcal{O}$$

for $i = 0, 1, j \geq 7$, and it is zero if $j \geq k+5$, here $*$ involves only $b_{i,j'-i}$ with $j' > j$ and elements in \mathcal{O} . Remember that $\sqrt{D} \in \mathcal{O}^*$. Apply $j = M+3$ to (3.8), we know that $b_{i,M+3-i} = 0$ for all $0 \leq i \leq 3$. Repeat the same argument, one shows that $b_{i,j-i} = 0$ for all $0 \leq i \leq 3$ and $j \geq k+5$. Now apply $j = k+4$ to (3.8), we get $p^m b_{i,k+4-i} \in \mathcal{O}$ for all $0 \leq i \leq 3$. Repeat the same argument and notice that the terms $*$ in (3.8) are in \mathcal{O} in each step (since all the $\delta_s^{\bullet, \bullet}$ and $b_{i,j'-i} \in \mathcal{O}$ if $j' > j$ in each step), one proves that $p^m b_{i,j-i} \in \mathcal{O}$ for all $0 \leq i \leq 3$ and $j \geq 7$. So $p^m b_{i,j-i} \in \mathcal{O} \cap \mathbb{Q}_q = \mathbb{Z}_q$ for all $0 \leq i \leq 3$ and $j \geq 7$.

(c) Consider

$$(3.9a) \quad \omega := p^m \left(x^k y^l dx - d \left(\sum_{i=0}^3 \sum_{j \geq 7-i}^{k+5} b_{i,j} x^j y^i \right) \right)$$

$$(3.9b) \quad = p^m \left(\sum_{i=1}^3 \sum_{j=0}^2 a_{i,j} x^j y^i dx + d \left(\sum_{i=0}^3 \sum_{j=0}^{6-i} b_{i,j} x^j y^i \right) \right)$$

From (b) and (3.9a), one knows that ω has integral coefficients, so we can choose $\varphi_1(x, y)$ and $\psi_1(x, y)$ in $\mathbb{Z}_q[x, y]$ such that $\omega = \psi_1 dx + \varphi_1 dy$. On the other hand, from (3.9b), one know that $\omega = \psi_2 dx + \varphi_2 dy$ for some φ_2 and ψ_2 in $\mathbb{Q}_q[x, y]$ with $\deg(\varphi_2) \leq 5, \deg(\psi_2) \leq 5$. Consider

$$(3.10) \quad \begin{aligned} f_y \omega &= f_y (\psi_1 dx + \varphi_1 dy) = (\psi_1 f_y - \varphi_1 f_x) dx \\ f_x \omega &= f_x (\psi_2 dx + \varphi_2 dy) = (\varphi_2 f_x - \psi_2 f_y) dy \end{aligned}$$

(using $f_x dx + f_y dy = df = 0$). Let $\lambda_i(x, y) := \psi_i f_y - \varphi_i f_x$. It is clear that $\lambda_1 \in \mathbb{Z}_q[x, y]$ and $\lambda_2 \in \mathbb{Q}_q[x, y]$ with $\deg(\lambda_2) \leq 8$. Using the defining equation f to reduce the degree of y in λ_i , we get $f_y \omega = \tilde{\lambda}_1 dx$ and $f_x \omega = -\tilde{\lambda}_2 dy$ with $\tilde{\lambda}_1 \in \mathbb{Z}_q[x, y], \deg_y(\tilde{\lambda}_1) \leq 3, \tilde{\lambda}_2 \in \mathbb{Q}_q[x, y], \deg(\tilde{\lambda}_2) \leq 8$ and $\deg_y(\tilde{\lambda}_2) \leq 3$. Since $(\tilde{\lambda}_1 - \tilde{\lambda}_2) dx = f_y \omega - f_x \omega = 0$ and $\deg_y(\tilde{\lambda}_1 - \tilde{\lambda}_2) \leq 3$, we have $\tilde{\lambda}_1 = \tilde{\lambda}_2$. This means that $f_y \omega = \tilde{\lambda} dx$ and $f_x \omega = -\tilde{\lambda} dy$ with $\tilde{\lambda} := \tilde{\lambda}_1 = \tilde{\lambda}_2$ which is in $\mathbb{Z}_q[x, y]$ of $\deg(\tilde{\lambda}) \leq 8$. By Corollary 3.9, there exist α and β in $\mathbb{Z}_q[x, y]$ with $\deg(\alpha) \leq 5$ and

$\deg(\beta) \leq 5$ such that $\alpha f_y + \beta f_x = 1$ in A . So $\omega = (\alpha f_y + \beta f_x) \omega = (\alpha \tilde{\lambda}) dx - (\beta \tilde{\lambda}) dy$. Notice that $\omega \equiv \sum_{i=1}^3 \sum_{j=0}^2 p^m a_{i,j} x^j y^i dx$, we can use the reduction of $(\alpha \tilde{\lambda}) dx - (\beta \tilde{\lambda}) dy$ to get the denominators of $p^m a_{i,j}$. Since $\deg(\alpha \tilde{\lambda}) \leq 13$ and $\deg(\beta \tilde{\lambda}) \leq 13$, we need only to know the denominators of the final reductions of $x^j y^i dx$ and $x^j y^i dy$ with $0 \leq i + j \leq 13$. Using $x^j y^i dy \equiv -j/(i + 1)x^{j-1}y^{i+1}dx$, the defining equation f , and $x^j dx \equiv 0$, we only need to consider the reductions of $x^j y^i dx$ with $1 \leq i \leq 3$ and $i + j \leq 13$, but if $p = 3$, the extra denominator 3 should be counted.

The reduction of $x^k y^l dx$ ($1 \leq l \leq 3$) using the reduction matrices $M_{i,j}$ in subsection 3.2 involve divisions by some of the following: $12(k + 2)b_4$, $(12k + 21)b_3$, $(k + 3)(a_2^2 - 4b_4)$, $(2k + 5)(a_1 a_2 - 2b_3)$, $384(k + 1)(k + 2)(k + 3)(k + 4)b_4^3(a_2^2 - 4b_4)$, $384(k + 2)(k + 3)(k + 4)(2k + 7)b_4^3(a_1 a_2 - 2b_3)$, $2(k + 4)(2k + 5)(4k + 3)(4k + 7)b_3^2 a_2^2$, $-(4k + 7)(4k + 11)(4k + 13)(4k + 15)b_3^3$, depending on each case, and the numbers $2, a_2, b_4, b_3, a_2^2 - 4b_4$ and $a_1 a_2 - 2b_3$ that we need to consider (depending on each case) are units of \mathbb{Z}_q . So in each step, we get extra denominators which are at most

$$p^{\lfloor \log_p(4j+15) \rfloor + \tau(p)},$$

here $\tau(3) = 5, \tau(5) = 3, \tau(p) = 1$ for $p = 7, 11, 13$ and $\tau(p) = 0$ if $p > 13$. Since we are concerned with $2 \leq j \leq 12$, we need at most 11 reduction steps, so the denominators of the reductions of $x^j y^i dx$ with $1 \leq i \leq 3$ and $i + j \leq 13$ are at most

$$p^{11 \cdot (\lfloor \log_p(63) \rfloor + \tau(p))}.$$

Hence $p^{m+\Delta+1} a_{i,j} \in \mathbb{Z}_q$. □

REMARK 3.7. Proposition 3.6 gives an upper bound for the denominators after a differential form (with integral coefficients) is reduced to the linear combination of the basis we found in Proposition 3.5. Along with the rate of convergence of the Frobenius F_p (see Corollary 4.3), one can determine how much p -adic precision we need to work with (and determine an integer N_3 such that one can work with modulo x^{N_3}), see Section 6. But one needs an upper bound for all the denominators that will appear during the computation (in the reduction step) in order to know how much precision of the reduction matrices $M_{i,j}$ are required and to have an analysis of the bit complexity. It turns out that one has a similar bound as in Proposition 3.6. The proof is completely similar.

THEOREM 3.8. *Let R be a field or a discrete valuation ring and \mathfrak{m} be the maximal ideal of R . Let $f_0, \dots, f_n \in R[x_1, \dots, x_n]$ with $\deg f_i = d_i$ and define*

$$\rho = d_0 + \dots + d_n - n - 1.$$

Denote the homogenization of f_i by f_i^h for $i = 0, \dots, n$. Assume that there is no point in $\mathbb{P}^n(\overline{R/\mathfrak{m}})$ satisfies $f_0^h = f_1^h = \dots = f_n^h = 0$. Then there exist polynomials $g_0, \dots, g_n \in R[x_1, \dots, x_n]$ with $\deg g_i \leq \rho + 1 - d_i$ for $i = 0, \dots, n$ such that

$$\sum_{i=0}^n g_i f_i = 1.$$

PROOF. This appears as Theorem 2 in Denef-Vercauteren [2]. □

COROLLARY 3.9. *There exist α and β in $\mathbb{Z}_q[x, y]$ with $\deg(\alpha) \leq 5$ and $\deg(\beta) \leq 5$ such that $\alpha f_y + \beta f_x = 1$ in A . Furthermore, one can find such α and β such that α has only odd degrees in y and β has only even degrees in y .*

PROOF. Apply 3.8 to $R = \mathbb{Z}_q$, $f_0 = f$, $f_1 = f_y$ and $f_2 = f_x$.

Apply 3.8 to $R = \mathbb{Z}_q$, $f_0 = f$, $f_1 = f_y$ and $f_2 = f_x$. If α and β don't satisfy the last property, consider the equality $\alpha(x, -y)f_y(x, -y) + \beta(x, -y)f_x(x, -y) = 1$ in A . From (3.1), it is clear that $f_y(x, -y) = -f_y(x, y)$ and $f_x(x, -y) = f_x(x, y)$. We thus have

$$\left(\frac{\alpha(x, y) - \alpha(x, -y)}{2}\right) f_y + \left(\frac{\beta(x, y) + \beta(x, -y)}{2}\right) f_x = 1,$$

which completes the proof. □

4. Lift of Frobenius

In this section, we describe a lift F_p of the absolute Frobenius endomorphism $\overline{F}_p : \overline{a} \rightarrow \overline{a}^p$ on the coordinate ring \overline{A} of $\overline{C}_{\text{aff}}$ to A^\dagger . This means that F_p is a \mathbb{Z}_p -algebra endomorphism on A^\dagger such that $\pi \circ F_p = \overline{F}_p \circ \pi$, where π is the reduction modulo p . The lift F_q of the q -th Frobenius endomorphism of \overline{A} is F_p^n , hence one can work with F_p for the purpose of computation.

Denote by σ the p -th power Frobenius endomorphism on \mathbb{F}_p and also its lift on \mathbb{Z}_p . Any lift F_p satisfies

$$F_p(x) \equiv x^p \pmod{p}, \quad F_p(y) \equiv y^p \pmod{p}, \quad F_p(f(x, y)) = 0.$$

From Corollary 3.9, we know that there exist α and β in $\mathbb{Z}_q[x, y]$ such that $\alpha f_y + \beta f_x = 1$. Define $\delta_y := \alpha^p$, $\delta_x := \beta^p$ and consider the equation

$$(4.1) \quad G(Z) := F_p(f(x, y)) = f^\sigma(x^p + \delta_x Z, y^p + \delta_y Z) = 0$$

in $A^\dagger[Z]$. Then $G(0) = f^\sigma(x^p, y^p) \equiv f^\sigma(x^\sigma, y^\sigma) = 0 \pmod{p}$. Also $G'(0) = f_y^\sigma(x^p, y^p)\delta_y + f_x^\sigma(x^p, y^p)\delta_x \equiv f_y^\sigma(x^\sigma, y^\sigma)\delta_y + f_x^\sigma(x^\sigma, y^\sigma)\delta_x \equiv f_y^p\delta_y + f_x^p\delta_x = f_y^p\alpha^p + f_x^p\beta^p = (f_y\alpha + f_x\beta)^p = 1 \pmod{p}$. Hence by Hensel's lemma, there is a unique solution of (4.1) in A^∞ . Use the following proposition and its corollary, this solution is in fact in A^\dagger . In fact, Corollary 4.3 below gives an explicit lower bound on the rate of convergence, which allows us (together with Proposition 3.5) to work with a finite and explicit p -adic precision.

LEMMA 4.1. *Let $H(Z) = \sum h_k(x)Z^k \in \mathbb{Z}_q[x][Z]$ and $\Delta_{1,k} = d_k := \deg(h_k)$. Assume $h_0(x) \equiv 0 \pmod{p}$ and $h_1(x) \equiv 1 \pmod{p}$. Let $0 \leq \Delta_{n,0} \leq \Delta_{n,1} \leq \dots \leq \Delta_{n,j} \leq \dots$ ($n \geq 1, j \geq 0$) with $\delta_j = \Delta_{1,j}$ be integers that satisfies the following conditions:*

1. $\Delta_{n+1,j} \geq \max\{\Delta_{n,j-l} + \delta_l \mid 0 \leq l \leq j\}$ for all $n \geq 0, j \geq 0$.
2. $\delta_0 \geq d_0$.
3. $\delta_j - \delta_{j-1} \geq d_1$ for all $j \geq 1$.
4. $\delta_{k-1+j} \geq \Delta_{k,j} + d_k$ for all $k \geq 2, j \geq 0$.

Then the unique solution $\alpha = \sum_{i=0}^\infty a_i x^i \in \mathbb{Z}_q\langle x \rangle$ has the property: $v_p(a_i) \geq j + 2$ if $i \geq \delta_j + 1$. One can always find such $\Delta_{n,j}$.

PROOF. Let $T := \{\sum_{i=0}^\infty a_i x^i \in \mathbb{Z}_q\langle x \rangle \mid v_p(a_i) \geq j + 2 \text{ if } i \geq \delta_j + 1\}$ be a subset of $\mathbb{Z}_q\langle x \rangle$, here $-1 = \delta_{-1} < 0 \leq \delta_0 \leq \delta_1 \leq \dots \leq \delta_j \leq \dots$ are integers which we will determine for which T satisfies some properties that are used in the proof. For each n , one can write $T^n = \{\sum_{i=0}^\infty a_i x^i \in \mathbb{Z}_q\langle x \rangle \mid v_p(a_i) \geq n + j + 1 \text{ if } i \geq \Delta_{n,j} + 1\}$ for some $-1 = \Delta_{n,-1} < 0 \leq \Delta_{n,0} \leq \Delta_{n,1} \leq \dots \leq \Delta_{n,j} \leq \dots$. A sufficient condition

for $T^n \subset T$ is $\Delta_{n+1,j} \geq \max\{\Delta_{n,j-l} + \delta_l \mid 0 \leq l \leq j\}$ for all n . Notice that T is closed under addition. In $\mathbb{Z}_q\langle x \rangle$, we can use Newton method:

$$\alpha_{i+1} = \alpha_i - \frac{H(\alpha_i)}{H'(\alpha_i)} = \alpha_i - H(\alpha_i) \left(1 + \sum_{k=1}^{\infty} (1 - H'(\alpha_i))^k \right).$$

Our goal is to determine a condition on δ_j for which the result in each iteration above is in T . We use induction: assume $\alpha_i \in T$ and to prove $\alpha_{i+1} \in T$. It is sufficient to show: $H(\alpha_i) \in T$ and $(1 - H'(\alpha_i))T \subset T$. $H(\alpha_i) = h_0(x) + h_1(x)\alpha_i + \sum_{k=2}^{\infty} h_k(x)\alpha_i^k$. Since $v_p(h_0(x)) \geq 1$, $h_0(x) \in T$ if $d_0 \leq \delta_0$. Since $h_1(x)\alpha_i = \alpha_i - (1 - h_1(x))\alpha_i$, $\alpha_i \in T$ and $v_p(1 - h_1(x)) \geq 1$, $h_1(x)\alpha_i \in T$ if $\delta_j - \delta_{j-1} \geq d_1$ for all $j \geq 1$. Similarly for $k \geq 2$, since $\alpha_i^k \in T^k$, $h_k(x)\alpha_i^k \in T$ if $\Delta_{k,j} + d_k \leq \delta_{k-1+j}$. In fact, the above conditions imply $(1 - h_1(x))T \subset T$ and $h_k(x)T^k \subset T$. Using this fact with $(1 - H'(\alpha_i))T = ((1 - h_1(x)) + \sum_{k=2}^{\infty} -kh_k(x)\alpha_i^{k-1})T$ and $\alpha_i^{k-1} \in T^{k-1}$, we know $(1 - H'(\alpha_i))T \subset T$. Hence $\alpha_{i+1} \in T$ and this implies the solution $\alpha \in T$.

For the existence $\Delta_{n,j}$, notice that the conditions 2, 3 and 4 are equivalent to: $\Delta_{1,n+1} \geq \max\{\Delta_{1,n} + d_1, \Delta_{k,n+2-k} + d_k \mid 2 \leq k \leq n+2\}$ for all $n \geq 0$. Suppose one has determined $\Delta_{n',j'}$ for all $n'+j' \leq n+1$. Use condition 1, one can determine $\Delta_{k,n+2-k}$ for all $2 \leq k \leq n+2$ (i.e for $\Delta_{n',j'}$ with $n'+j' \leq n+2$ and $n' \geq 2$). Finally, one determines $\Delta_{1,n+1}$. Therefore, one determines all the $\Delta_{n',j'}$ with $n'+j' \leq n+2$. This shows that one can find $\Delta_{n,j}$ recursively. \square

LEMMA 4.2. *Suppose $\deg(h_k(x)) \leq (k+1)d$ in Lemma 4.1. Then $\Delta_{i,j} := (i+4j)d$ satisfy the conditions in Lemma 4.1 and all the inequalities are equalities. In particular, $\delta_j = (4j+1)d$.*

PROOF. This follows by induction in i and j . \square

COROLLARY 4.3. *There exists a lift F_p of the absolute Frobenius endomorphism $\bar{\alpha} \rightarrow \bar{\alpha}^p$ on the coordinate ring \bar{A} of \bar{C}_{aff} to A^\dagger such that $F_p(x) = x^p + \delta_x Z_0$ and $F_p(y) = y^p + \delta_y Z_0$ with $Z_0 = \sum_{i,j} a_{i,j} x^i y^j$, $a_{i,j} \in \mathbb{Z}_q$ and $\text{ord}_p(a_{i,j}) > \frac{i+j}{16p}$. Also the coefficient of $x^j y^i$ in $F_p(y)$ and $F_p(x)$ has p -adic order $> \frac{i+j}{16p}$ if $i+j \neq p$. Finally, $F_p(x^k y^l dx) = \sum_{i=1}^3 \sum_j b_{i,j} x^i y^j dx$ with $\text{ord}_p(b_{i,j}) > \frac{i+j}{16p} - 4$.*

PROOF. Using Corollary 3.9 and equation (4.1), one can apply $d = 4p$ to Lemma 4.2. \square

THEOREM 4.4. *There exists a lift of Frobenius F_p on A^\dagger which commutes with the involution $\tau : y \rightarrow -y$ and has the rate of convergence in Corollary 4.3.*

PROOF. We choose α and β such that α has only odd degrees in y and β has only even degrees in y as in Corollary 3.9. Since $\delta_y = \alpha^p$ and $\delta_x = \beta^p$, they have the same property as α and β . For solving $G(Z) = 0$ by Newton's method, we use Lemma 4.1 with $H = G = f^\sigma(x^p + \delta_x Z, y^p + \delta_y Z)$. It is clear that G has only even degrees in y , hence so does the solution Z_0 . From this, it is clear that the lift of Frobenius $F_p : A^\dagger \rightarrow A^\dagger$ commutes with the involution τ . \square

5. Quotient by Automorphism

We have study $H_{Mw}^1(C_{\text{aff}}/\mathbb{F}_q)$. In this section, we consider the quotient of C by the automorphism $\tau : Y \rightarrow -Y$. We denote the quotient map by $\pi : C \rightarrow E :=$

$C/\langle\tau\rangle$. One can show that $C/\langle\tau\rangle$ has genus 1 either by Riemann-Hurwitz genus formula or from the affine equation directly, using the fact that C_{aff} is stable under τ and $C_{\text{aff}}/\langle\tau\rangle$ is smooth, hence the notation E is justified. The affine part E_{aff} of E is $C_{\text{aff}}/\langle\tau\rangle$, which has the defining equation: $v^2 + \bar{g}(u)v + \bar{h}(u) = 0$. We have $C_{\text{aff}} \xrightarrow{\pi} E_{\text{aff}}$, $(x, y) \rightarrow (x, y^2)$, and the corresponding map on the coordinate ring is $\pi^* : u \rightarrow x, v \rightarrow y^2$.

Our goal is to study the followings: $H^1_{MW}(E_{\text{aff}}/\mathbb{F}_q)$, the induced map $\pi^* : H^i_{MW}(E_{\text{aff}}/\mathbb{F}_q) \rightarrow H^i_{MW}(C_{\text{aff}}/\mathbb{F}_q)$ and its interplay with Frobenius endomorphism. Since $\tilde{C}_{\text{aff}} \xrightarrow{\tilde{\pi}} \tilde{E}_{\text{aff}}$, $(x, y) \rightarrow (x, y^2)$ lifts π , here \tilde{E}_{aff} is the lift of E_{aff} , whose defining equation is $v^2 + g(u)v + h(u) = 0$, we can study $H^i_{MW}(E_{\text{aff}}/\mathbb{F}_q) \xrightarrow{\pi^*} H^i_{MW}(C_{\text{aff}}/\mathbb{F}_q)$ by $H^i_{dR}(\tilde{E}_{\text{aff},\xi}/\mathbb{Q}_q) \xrightarrow{\tilde{\pi}^*} H^i_{dR}(\tilde{C}_{\text{aff},\xi}/\mathbb{Q}_q)$. For $i \neq 1$, these are isomorphisms. For $i = 1$, since $\{\tilde{\pi}^*(u^j v du) = x^j y^2 dx \mid 0 \leq j \leq 1 \text{ (resp. } 0 \leq j \leq 2)\}$ are linear independent in Case 1 and Case 3 (resp. in Case 2 and Case 4), one sees that $\{u^j v du \mid 0 \leq j \leq 1 \text{ (resp. } 0 \leq j \leq 2)\}$ are linear independent. Let δ_E be the number of points at infinity of E . We have $\delta_E = 1, 2, 1, 2$ in each case, hence $\dim_{\mathbb{Z}_q} H^1_{dR}(\tilde{E}_{\text{aff},\xi}/\mathbb{Q}_q) = 2 \cdot g_E - 1 + \delta_E = 2, 3, 2, 3$. This shows that $\{u^j v du \mid 0 \leq j \leq 1 \text{ (resp. } 0 \leq j \leq 2)\}$ is the basis of $H^1_{dR}(\tilde{E}_{\text{aff},\xi}/\mathbb{Q}_q)$ which is isomorphic via $\tilde{\pi}^*$ to the subspace V of $H^1_{dR}(\tilde{C}_{\text{aff},\xi}/\mathbb{Q}_q)$ generated by $\{x^j y^2 dx \mid 0 \leq j \leq 1 \text{ (resp. } 0 \leq j \leq 2)\}$.

As in Section 4, there is a lift $F_{q,E} : A^\dagger_E \rightarrow A^\dagger_E$ of the Frobenius endomorphism $\bar{F}_{q,E}$ on the coordinate ring \bar{A}_E of E . The left diagram below is not necessary commutative, but its reduction mod p is commutative

$$\begin{array}{ccc}
 A^\dagger & \xleftarrow{\tilde{\pi}^*} & A^\dagger_E \\
 F_q \downarrow & & \downarrow F_{q,E} \\
 A^\dagger & \xleftarrow{\pi^*} & A^\dagger_E
 \end{array}
 \qquad
 \begin{array}{ccc}
 \bar{A} & \xleftarrow{\pi^*} & \bar{A}_E \\
 \bar{F}_q \downarrow & & \downarrow \bar{F}_{q,E} \\
 \bar{A} & \xleftarrow{\pi^*} & \bar{A}_E
 \end{array}$$

Here $A^\dagger \xleftarrow{\tilde{\pi}^*} A^\dagger_E$ is the natural lift of homomorphism $A \xleftarrow{\tilde{\pi}^*} A_E$ on the coordinate rings which corresponds to the morphism $\tilde{\pi} : \tilde{C}_{\text{aff},\xi}/\mathbb{Q}_q \rightarrow \tilde{E}_{\text{aff},\xi}/\mathbb{Q}_q$, so the reduction of $A^\dagger \xleftarrow{\tilde{\pi}^*} A^\dagger_E$ modulo p is just the natural homomorphism on the coordinate rings of $C_{\text{aff}} \xrightarrow{\pi} E_{\text{aff}}$. Since $\bar{F}_q \circ \pi^* = \pi^* \circ \bar{F}_{q,E}$, we know that

$$\begin{array}{ccc}
 H^i_{MW}(C_{\text{aff}}/\mathbb{F}_q) & \xleftarrow{\pi^*} & H^i_{MW}(E_{\text{aff}}/\mathbb{F}_q) \\
 F_{q,*} \downarrow & & \downarrow F_{q,E,*} \\
 H^i_{MW}(C_{\text{aff}}/\mathbb{F}_q) & \xleftarrow{\pi^*} & H^i_{MW}(E_{\text{aff}}/\mathbb{F}_q)
 \end{array}$$

So the point counting on E_{aff} is the same as computing on the subspace of $H^1_{MW}(C_{\text{aff}}/\mathbb{F}_q)$ generated by $\{x^j y^2 dx \mid 0 \leq j \leq 1 \text{ (resp. } 0 \leq j \leq 2)\}$.

From Lefschetz fixed point formula (Theorem 2.3), we have

$$\begin{aligned}
 \#C_{\text{aff}}(\mathbb{F}_{q^r}) &= \text{Tr}((qF_{q,*}^{-1})^r | H^0_{MW}(C_{\text{aff}})) - \text{Tr}((qF_{q,*}^{-1})^r | H^1_{MW}(C_{\text{aff}})) \\
 \#E_{\text{aff}}(\mathbb{F}_{q^r}) &= \text{Tr}((qF_{q,*}^{-1})^r | H^0_{MW}(E_{\text{aff}})) - \text{Tr}((qF_{q,*}^{-1})^r | H^1_{MW}(E_{\text{aff}}))
 \end{aligned}$$

Let $P_E(X) = (X - \beta_1)(X - \beta_2)$ be the Weil polynomial of E and $S_r(E) := \beta_1^r + \beta_2^r$. Then $\#E_{\text{aff}}(\mathbb{F}_{q^r}) = q^r + 1 - S_r(E) - \delta_E$, here δ_E is the number of points at infinity of E . Use $\#C_{\text{aff}}(\mathbb{F}_{q^r}) = \#C(\mathbb{F}_{q^r}) - \delta_C$, we get

$$\#C(\mathbb{F}_{q^r}) = q^r + 1 - S_r(E) - \text{Tr}((qF_{q,*}^{-1})^r|V) + (\delta_C - \delta_E),$$

here V is the subspace of $H_{MW}^1(C_{\text{aff}}/\mathbb{F}_q)$ generated by $\{x^j y^2 dx \mid 0 \leq j \leq 1 \text{ (resp. } 0 \leq j \leq 2)\}$, whose dimension is $4 + \delta_C - \delta_E$. This implies that the Weil polynomial $P(X)$ of C/\mathbb{F}_q equals to $P_E(X)Q_V(X)(X - 1)^{-(\delta_C - \delta_E)}$, here $Q_V(X)$ is the characteristic polynomial of $qF_{q,*}^{-1}$ acts on V . The characteristic polynomial $P_V(X)$ of $F_{q,*} = q \cdot (qF_{q,*}^{-1})^{-1}$ is $Q_V(X)(X - q)^{\delta_C - \delta_E}(X - 1)^{-(\delta_C - \delta_E)}$, so

$$P(X) = P_E(X)P_V(X)(X - q)^{-(\delta_C - \delta_E)}.$$

SUMMARY 5.1. The Weil polynomial $P(X)$ of C/\mathbb{F}_q is equal to

$$P_E(X)P_V(X)(X - q)^{-(\delta_C - \delta_E)},$$

where $P_E(X)$ is the Weil polynomial of E/\mathbb{F}_q and $P_V(X)$ is the characteristic polynomial of $F_{q,*}$ on V .

6. The Algorithm

In order to compute $P_V(X)$, one needs to compute $P_V(X)$ with a precision $N_1 = \lfloor \log_p 30 + 2n \rfloor + 1$ with $n = \log_p q$, which is determined by the Weil bound. Due to the fact that the matrix M_p of the Frobenius action $F_{p,*}$ may have denominators, we need M_p with a precision $N_2 := N_1 + (6n - 1)c$ with $c = \lfloor c_1 + \log_p(c_1 + \log_p(2c_1)) \rfloor + 1$ and $c_1 = 6 + \log_p 80 + \Delta$. From this, we only need to compute (for $1 \leq l \leq 3$ and $0 \leq k \leq 2$) $Z_0, F_p(x), F_p(y)$ and $F_p(y^l x^k dx)$ modulo (x^{N_3}, p^{N_4}) with $N_3 = \lfloor 16p(c_2 + \log_p(2c_2)) \rfloor + 1, N_4 = \lfloor N_2 + c_1 + \log_p(c_2 + \log_p(2c_2)) \rfloor + 1$ and $c_2 = 6 + \log_p 80 + \Delta + N_2$. Finally, the above discussion is based on the reduction matrices $M_{i,k}$ ($1 \leq i \leq 3$) introduced in 3.2. But since one can only work with an approximation of $M_{i,k}$, one need $M_{i,k}$ modulo p^{N_5} with a slightly higher precision $N_5 = N_4 + 8 \lfloor \log_p N_3 \rfloor + 14$. We have $N_3 = O(pn), N_4 = O(n)$ and $N_5 = O(n)$. We work in \mathbb{Z}_q/p^{N_5} , whose elements can be stored in $O(n^2 \log p)$ space and the arithmetic on it could be done in $\tilde{O}(n^2 \log p)$ bit operations. This gives the algorithm:

Algorithm

- Step 1.** Compute α and β in Corollary 3.9 modulo p^4
- Step 2.** Compute Z_0 in Corollary 4.3 using Newton's method, then $F_p(x^k y^l dx)$ for $l = 1, 3$ and $0 \leq k \leq 2$, all of them are modulo (x^{N_3}, p^{N_4}) .⁵
- Step 3.** Use reduction matrices $M_{i,j}$ ($1 \leq i \leq 3, 2 \leq j \leq N_3$) to reduce $F_p(x^k y^l dx)$ and get M_p .
- Step 4.** Compute $M_q = M_p M_p^\sigma \cdots M_p^{\sigma^{n-1}}$ by repeated squaring.
- Step 5.** Finally, compute the characteristic polynomial $P_V(X)$ modulo P^{N_1} .

³More precisely, with p -adic precision N_4 but with denominators at most $p^{(N_5 - N_4)}$.

⁴In Corollary 3.9, we only need $\alpha f_y + \beta f_x \equiv 1$ modulo p in order to compute the lift of Frobenius.

⁵In the proof of Lemma 4.1, we showed that the results during the Newton's iteration all have the same rate of convergence as in Corollary 4.3, so we can work modulo x^{N_3} during the Newton's iteration.

THEOREM 6.1. *The above algorithm requires $\tilde{O}(n^3p)$ bit operations.*

PROOF. Step 1 consists of solving a system of linear equations over \mathbb{F}_q of size at most 16. Hence it requires $\tilde{O}(n^2 \log p)$ bit operations. Step 2 requires $O(\log N_4)$ Newton's iterations, and each iteration requires $\tilde{O}(N_3 n^2 \log p)$ bit operations. Hence this step requires $\tilde{O}(pn^3)$ bit operations. Step 3 requires $O(N_3)$ operations in \mathbb{Z}_q/p^{N_5} , hence $\tilde{O}(pn^3)$ bit operations. Step 4 requires $O(\log n)$ squarings and the application of the lift of the p -th power Frobenius $\sigma : \mathbb{Z}_q \rightarrow \mathbb{Z}_q$ modulo p^{N_4} on matrices of size 6×6 . Squaring requires $\tilde{O}(n^2 \log p)$ bit operations. For σ , we use Newton's method which needs to evaluate a polynomial of degree n with coefficients in \mathbb{Z}_q/p^{N_4} , which requires $O(n)\tilde{O}(n^2 \log p)$ bit operations. Hence we need $\tilde{O}(n^3 \log p)$ bit operations in this step. Step 5 requires $\tilde{O}(n^2 \log p)$ bit operations. Hence the algorithm requires $\tilde{O}(n^3p)$ bit operations. \square

If one works directly on $H_{MW}^1(C_{\text{aff}})$ and denote the precision needed by N'_i , then $N_i \approx \frac{2}{3}N'_i$. Also the matrix M_p is of size 6×6 and M'_p is of size 9×9 . From these, we give a comparison of speed. Step 2 is reduced by a factor of $(\frac{2}{3})^2 \approx 0.45$. In Step 3, we have 6 differential forms $F_p(x^k y^l dx)$, $l = 1, 3$ and $0 \leq k \leq 2$, to reduce. This contributes a factor of $\frac{4}{5}$. (It is $\frac{4}{5}$ instead of $\frac{2}{3}$ because the reductions of $F_p(x^k y^2 dx)$ involve fewer operations than the reductions of $F_p(x^k y^1 dx)$ and $F_p(x^k y^3 dx)$. See the reduction matrices in 3.2.) Since each of these $F_p(x^k y^l dx)$ is computed modulo (x^{N_3}, p^{N_4}) in Step 2, we work with smaller powers on x and fewer p -adic precision in Step 3. This means that we have fewer reduction steps and the basic arithmetic operations are faster, which contribute a factor of $(\frac{2}{3})^2$. So Step 3 is reduced by a factor of $\frac{4}{5} \cdot (\frac{2}{3})^2 \approx 0.36$. Step 4 is reduced by a factor at least of $(\frac{2}{3})^3 \approx 0.3$, due to the smaller size of M_p and fewer precision.

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